Global Uniform Synchronization With Estimated Error Under Transmission Channel Noise

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***Abstract*—This paper investigates the problem of estimating synchronization errors and its application to global uniform synchronization with an estimated error bound for the master–slave chaos synchronization scheme via linear control input, which is possibly subject to disturbances by unknown but bounded channel noise and time-delay. Based on Lyapunov function, Razumikhin technique, nonlinear parametric variation, and input-to-state stability (ISS) theory, estimation formulas of synchronization errors with or without time-delays but with noise in transmission channel (TC) are derived. By using the error estimation formula, the maximal upper bound for time-delays is also obtained. These formulas can be used to design a control gain matrix which forces the synchronization error to the minimal value. Meanwhile, theoretical discussion is made by comparing Lyapunov–Krasovskii function method with respect to time-delays in TC. After the theoretical analysis, some representative examples and their numerical simulations are given for illustration.**

***Index Terms*—Chaotic synchronization, error bound, global exponential synchronization, global uniform synchronization, ISS, Razumikhin technique, synchronization error, time-delay.**

I. Introduction

CHAOTIC synchronization has attracted increasing attention due to its great potential in applications such as biological systems (neural network functioning, brain activities, heartbeat regulation, etc.), oscillator design, vibrating wave generation, mechanical resonance, spatiotemporal pattern formation, and secure communications. Many strategies and methods for chaotic synchronization have been developed since the early 1990s [1]–[9], among which feedback control is especially attractive and effective. Many results on chaotic synchronization via feedback control are now available in the literature [10]–[13].

It has been noticed, however, that when chaotic synchronization is applied in engineering applications such as communications, the major shortcoming of many recently proposed chaos-based communication schemes is their susceptibility to noise and time-delays through the transmission channel (TC).

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Channel noise may be present in different forms and usually destroys good properties of the synchronized systems. Moreover, time delays occur commonly in synchronization scheme and other practical engineering systems due to the congestion of the network traffic and the fact that the switching speed of the hardware and circuit implementation is finite. Just as channel noise, time-delays also leads to failure of a synchronization scheme and instability of stable systems. When there exist channel noise and time-delays in TC, it's almost impossible to synchronize completely master–slave systems. It is therefore important to synchronize master–slave systems within an error bound. Although robust synchronization in the case of parameter mismatch in nonidentical Lur'e chaotic systems and dynamical networks have been investigated [15]–[22], [30]–[33], there are very few theoretical results for chaotic synchronization in which the channel is disturbed by unknown noise and time-delays. To the best of our knowledge, no general theory has been formulated for the estimation of synchronization errors subject to unknown channel noise and time-delays.

It should be noted that, in the literature, Lyapunov–Krasovskii function method is often used to analyze the delay-dependent stability properties for time-delayed systems. Recently, in [34]–[37], delay-dependent *asymptotical synchronization* criteria for Lur'e systems are established by using this kind of Lyapunov function. In this paper, we employ Razumikhin technique and input-to-state stability (ISS) theory to investigate the synchronization issue with respect to TC noises and time-delays. Compared with Lyapunov–Krasovskii function method, Razumikhin technique has advantage that, when dealing with time delays, the Lyapunov function is not required to be decreasing on the whole state space. The Razumikhin technique has also been applied successfully by various authors to study of stability problem for time-delayed systems, see, for instance, [38]–[44]. There has difficulties in using Lyapunov–Krasovskii function to analyze ISS or synchronization issue under TC disturbances and time-delays. The reason is that Lyapunov–Krasovskii function often has complex structure (it is a sum of a quadric positive definite function and several nonnegative functions with integration) and thus it is hard to derive ISS properties and the error estimation formula of error system. Moreover, most results obtained by using Lyapunov–Krasovskii function are asymptotic stability, not exponential stability. Thus, even for synchronization scheme with no noise but with time-delays in TC, it is also hard to derive the *exponential synchronization* criterion by using Lyapunov–Krasovskii function.

In this paper, a chaotic synchronization problem subject to noise disturbances and time-delays in TC is considered for chaotic systems. More precisely, the estimation problem of synchronization errors is studied, where the channel is subject to unknown but bounded noise disturbances and time-delays, for a setting of master–slave chaotic synchronization. We aim to investigate and estimate the synchronization error and achieve exponential synchronization when there is no TC noises. The exponential synchronization scheme has an obvious advantage over other synchronization schemes, in which the synchronization speed and synchronization time can be estimated easily. By employing the methods of Lyapunov function, Razumikhin technique and ISS theory for nonlinear systems, estimation formulas of synchronization errors with or without time-delays in TC are derived. The maximal above bound for time-delays is also estimated such that it can be used to design a linear feedback control input which forces the synchronization error to the minimal value. Moreover, a global exponential synchronization criteria is derived for the noise-free but with time-delays situation. Meantime, in the case of time-delays in TC, theoretical discussion is made by comparing Lyapunov–Krasovskii function method. Examples and simulations are given to illustrate the theoretical results.

The rest of this paper is organized as follows. In Section II, the master–slave synchronization scheme for chaotic systems is first formulated, subject to unknown but bounded noise disturbances and time-delays in TC. The approach to achieve synchronization via linear control input is also proposed in this section. Then, in Section III, estimation formulas of synchronization errors with or without time-delays but with channel noise in TC are established respectively. In Section IV, some representative examples are given for the purpose of illustration.

II. Preliminaries and Problem Formulation

In the sequel, R^n denotes the n-dimensional Euclidean space and $R_+ = [0, +\infty)$. A function $\gamma : R_+ \to R_+$ is of class-$\mathcal{K}$ ($\gamma \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and unbounded. A continuous function $\beta : R_+ \times R_+ \to R_+$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$. A function $V : R_+ \times R^n \to R_+$ is of class-$\mathcal{V}$ if $V(t, x)$ is continuous and continuously differentiable on $R_+ \times R^n$. Let $\|\cdot\|$ stand for the Euclidean norm in R^n. Given a constant $\tau > 0$, define by $\|\phi\|_\tau = \sup_{-\tau \leq s \leq 0} \|\phi(s)\|$, and for all $t \in R_+$ and $\psi \in C([-\tau, +\infty), R_+)$, define by $\|\psi_\tau(t)\| = \sup_{t-\tau \leq s \leq t} \|\psi(s)\|$, $\|\psi_\tau\|_\infty = \sup_{t \in R_+} \|\psi_\tau(t)\|$, and $\|\psi\|_\infty = \sup_{t \in R^+} \|\psi(t)\|$.

In applications, when the signals are transmitted from the master system to the slave system, the TC is unavoidably contaminated with noise. Here, we formulate this type of chaotic synchronization scheme as follows.

The master system

$$M : \begin{cases} \dot{x}(t) &= Ax(t) + \varphi(t, x(t)) \\ p(t) &= Hx(t) \\ \hat{p}(t) &= Kp(t) \end{cases} \tag{1}$$

where $x, p \in R^n, A \in \mathbb{R}^{n \times n}, \varphi \in C[R_+ \times R^n, R^n]$, $H \in R^{n \times n}$ is a nonsingular output gain matrix, and $K \in R^{n \times n}$ is a control gain matrix.

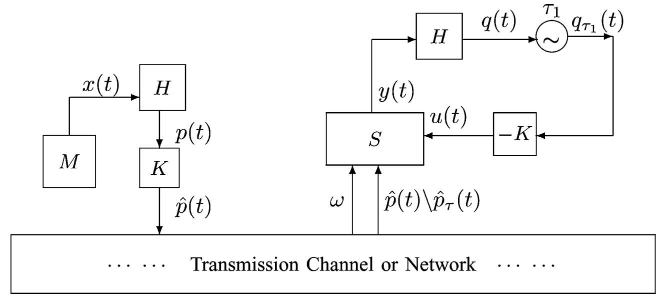

Fig. 1. The synchronization framework subject to noise and delay.

The slave system

$$S : \begin{cases} \dot{y}(t) &= Ay(t) + \varphi(t, y(t)) + \hat{p}(t) \backslash \hat{p}_\tau(t) + u(t) + \omega \\ q(t) &= Hy(t) \\ \hat{q}(t) &= Kq(t) \end{cases} \tag{2}$$

where if there is time-delay τ in TC or network, the signal $\hat{p}_\tau(t) = Kp(t - \tau(t))$ is transmitted to slave system S at time t, otherwise, signal $\hat{p}(t)$ is transmitted to slave system S; $u(t)$ is the control input which is in one of the following forms:

$$C : u(t) = \begin{cases} u_0(t) = -\hat{q}(t) \\ u_{\tau_1}(t) = -\hat{q}_{\tau_1}(t) \\ u_\tau(t) = -\hat{q}_\tau(t) \end{cases} \tag{3}$$

where $\tau(t)$ is the time-delay in TC from the master system M to the slave system S at the time t, and $\tau_1(t)$ the time-delay in slave system at time t, satisfying $0 \leq \tau_1(t), \tau(t) \leq \tau$ for all $t \in R_+$ and some constant $\tau \geq 0$; $\hat{q}_s(t) = Kq_s(t)$, $q_s(t) = q(t - s)$, where $s = \tau_1$ or $s = \tau$; and $\omega(t)$ the noise at time t, which is assumed to be unknown but bounded, i.e., $\|\omega(t)\| \leq \|\omega\|_\infty < \infty$.

Remark 2.1: Fig. 1 depicts the entire setting with noise disturbances with or without time-delays. In this synchronization scheme, the control gain matrix K is designed to put on the output part of both systems (master and slave systems) in order to avoid enlarging the noise ω. In the literature, the linear error feedback scheme is often used to achieve the synchronization. In this case, if there exists noise ω in TC, then, it leads to $K\omega$ and hence the noise ω is enlarged by the control gain K.

Define the synchronization error as $e(t) = y(t) - x(t)$. If there is no time-delay in TC, then, one has an error dynamical system of the form

$$\dot{e}(t) = Ae(t) + g(t, e, y) + \hat{p}(t) + u_0(t) + \omega(t) \tag{4}$$

where $g(t, e, y) = \varphi(t, y) - \varphi(t, y - e)$. Denote by $e(t) = e(t, t_0, e_0)$ the solution of (4) such that $e(t_0) = e_0 = x_0 - y_0$, where x_0 and y_0 are the initial condition of system (1) and (2), respectively.

If there exist time-delays for signal in TC, then, one can rewrite error system (4) as follows:

$$\dot{e}(t) = Ae(t) + g(t, e, y) + \hat{p}_\tau(t) + u(t) + \omega(t) \tag{5}$$

where $u(t) = u_0(t)$, $u(t) = u_{\tau_1}(t)$ or $u(t) = u_\tau(t)$, for all $t \in R_+$. Denote by $e(t) = e(t, t_0, \phi)$ the solution of (5) such that $e(t_0) = \phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are the initial condition of system (1) and (2), respectively, satisfying $\phi_1, \phi_2 \in C([-\tau, 0], R^n)$.

Definition 2.1: The noise-free synchronization scheme (1)–(3) is said to achieve global exponential synchronization if, for any initial condition x_0, y_0, the trivial solution $e = 0$ of the error dynamical system (4) is exponentially stable, in the sense that there exist two positive numbers, $\alpha > 0, M \geq 1$, such that

$$\|e(t)\| \leq M\|x_0 - y_0\| e^{-\alpha(t-t_0)}, \quad t \geq t_0 \tag{6}$$

for the case of time-delay with the initial condition $\phi_1, \phi_2 \in C([-\tau, 0], R^n))$, the noise-free synchronization scheme (1)(1)–(3) is said to achieve global exponential synchronization if

$$\|e(t)\| \leq M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \geq t_0 \tag{7}$$

where $\phi = \phi_2 - \phi_1$.

Definition 2.2: The synchronization scheme (1)–(3) with noise in TC is said to achieve global uniform synchronization with an error bound ϵ if for any initial condition there exists a $T \geq t_0$ such that $\|e(t)\| \leq \epsilon$ for all $t \geq T$, i.e., $\lim_{t\to\infty} \sup \|e(t)\| \leq \epsilon$.

Assumption 2.1: The noise may be unknown but satisfies $\omega(0) = 0$, $\|\omega\|_\infty < +\infty$ and $\|\omega_\tau\|_\infty < +\infty$.

Assumption 2.2: There exists a positive real constant $\delta > 0$ such that for any initial condition $x_0 \in R^n$ there exists a time $T(x_0)$ satisfying

$$x(t, t_0, x_0) \in D \triangleq \{x \,|\, \|x(t)\| \leq \delta, \quad t \geq T\}.$$

Assumption 2.3: φ satisfies the Lipschitz condition uniformly with respective to the second variable, namely, for some $L > 0$, such that

$$\|\varphi(t, x) - \varphi(t, y)\| \leq L\|x - y\|, \quad \text{for all } t \geq t_0. \tag{8}$$

Remark 2.2: Many chaotic systems can be rewritten in form of Lur'e system: $f(t, x) = Ax + \varphi(t, x)$. From the boundedness of chaotic system, i.e., $\|x\|_\infty < \infty$, function $\varphi(t, \cdot)$ often satisfies (8) for some $L > 0$ and any $t \in R_+$. Hence, assumption Assumptions 2.2–2.3 is rational.

The main objective of this paper is to estimate the bound ϵ of the synchronization error $\|e(t)\|$ and hence investigate global uniform synchronization with ϵ, as specified in Definition 2.2, when the TC is contaminated with unknown but bounded noise and possible time-delays. To do so, we need the following preliminaries.

Lemma 2.1: [26] Let $\nu > 0$ and $\alpha \in \mathcal{K}$. If $m(t) \geq \nu$ implies that the Dini derivative $D^+m(t) \leq -\alpha(m(t))$, then there exists $\beta \in \mathcal{KL}$ (independent of ν) with $\beta(s, 0) \geq s$ such that

$$m(t) \leq \max\{\beta(m(t_0), t - t_0), \nu\}. \tag{9}$$

The following is a key lemma in obtaining our result given in Section III-B. Its proof is similar to the proof of Razumikhin-type exponential stability theorems for impulsive time-delay systems (see [23]).

Lemma 2.2: For a time-delay system $\dot{x} = f(t, x, x_\tau)$ (see [23]), assume that the uniqueness and existence of solution for the system is guaranteed and there exists a function $V(t, x) \in \mathcal{V}$ and there exist constants $p > 0, q > 1, c_1 > 0, c_2 > 0, \lambda > 0$ such that the following conditions hold:

i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$;

ii) $D^+V(t, \varphi(0)) \leq -\lambda V(t, \varphi(0))$, for all $t \geq t_0$, whenever $qV(t, \varphi(0)) \geq V(t + s, \varphi(s))$ for $s \in [-\tau, 0]$.

Then system $\dot{x} = f(t, x, x_\tau)$ is global exponential stable and its Lyapunov exponent should not be greater than $-\alpha/p$, where $\alpha = \min\{\lambda, (\ln q)/\tau\}$.

III. Main Results

In this section, the estimation problem of synchronization errors is studied in two parts. The first part aims to investigate the case with no time-delays in TC while the second part studies the case with time-delays. The synchronization error formulas under which the uniformly synchronizing with an error bound achieves are derived.

A. Error Estimation and Uniform Synchronization for the Case With No Time-Delays in TC

Theorem 3.1: Let $u = u_0 = -Kq(t)$, where K is the control gain matrix, and Assumptions 2.1–2.2 hold and also assume that there exist a positive definite and symmetric matrix P and a positive constant $\alpha > 0$ such that

$$PA_K + A_K^T P + 2L\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}P \leq -\alpha P \tag{10}$$

where $A_K = A - KH$.

Then, the noise-free synchronization scheme (1)–(3) achieves global exponentially synchronizing and the actual synchronization scheme (1)–(3) with noise achieves global uniform synchronization with an error bound $2/\alpha\sqrt{(\lambda_{\max}(P))/(\lambda_{\min}(P))}\|\omega\|_\infty$.

Proof: Let $V(e) = e^T Pe$ and $e(t, t_0, e_0)$ denote the solution of (4) passing through the initial point (t_0, e_0). By (10), one has

$$\begin{aligned} &D^+V(e)\,|_{(4)} \\ &\quad= 2e^T(Ae + g(t, e, y) - KHe) + 2e^T P\omega(t) \\ &\quad= e^T\left[PA_K + A_K^T P\right]e + 2e^T Pg(t, e, y) + 2e^T P\omega(t) \\ &\quad\leq e^T\left[PA_K + A_K^T P + 2L\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}P\right]e + 2e^T P\omega(t) \end{aligned} \tag{11}$$

$$\begin{aligned} &\quad\leq -\alpha V(e) + \mu_1 V(e) + \mu_1^{-1}\|P\|\|\omega(t)\|^2 \\ &\quad= -(\alpha - \mu_1)V(e) + \mu_1^{-1}\|P\|\|\omega\|_\infty^2 \end{aligned} \tag{12}$$

where μ_1 is some positive constant satisfying $0 < \mu_1 < \alpha$.

Hence, if there exists no noise in the TC, i.e., $\omega(t) \equiv 0$, then by (11), we have

$$V(e(t)) \leq V(e_0)e^{-\alpha(t-t_0)}, \quad (\omega(t) \equiv 0, t \geq t_0) \tag{13}$$

which implies that

$$\|e(t)\| \leq \sqrt{(\lambda_{\max}(P))/(\lambda_{\min}(P))}\|e_0\|e^{-(\alpha)/(2)(t-t_0)}$$

and hence the noise-free synchronization scheme (1)–(3) achieves global exponential synchronization.

On the other hand, if there exists noise $\omega(t) \neq 0$ in the TC, then, from (12), when $V(e(t)) \geq (\mu_1^{-1}\|P\|\|\omega\|_\infty^2)/(\mu(\alpha-\mu_1))$, where $\mu < 1$, then

$$D^+V(e)\,|_{(4)} \leq -(1-\mu)(\alpha-\mu_1)V(e). \tag{14}$$

By Lemma 2.1, we get that there exists a function $\beta \in \mathcal{KL}$ with $\beta(s,0) \geq s$ such that

$$V(e) \leq \max\left\{\beta(V(e_0), t-t_0), \frac{\mu_1^{-1}\|P\|\|\omega\|_\infty^2}{\mu(\alpha-\mu_1)}\right\}. \tag{15}$$

Thus, we have

$$\begin{aligned}\|e(t)\| &\leq \max\left\{\frac{\sqrt{\beta(V(e_0), t-t_0)}}{\lambda_{\min}(P)}, \left(\frac{\mu_1^{-1}\|P\|\|\omega\|_\infty^2}{\mu(\alpha-\mu_1)\lambda_{\min}(P)}\right)^{\frac{1}{2}}\right\}\\ &\leq \max\left\{\tilde{\beta}(\|e_0\|, t-t_0), \left(\frac{\mu_1^{-1}\|\omega\|_\infty^2}{\mu(\alpha-\mu_1)\lambda(P)}\right)^{\frac{1}{2}}\right\} \end{aligned}\tag{16}$$

where $\tilde{\beta}(s,t) = (\sqrt{\beta(\lambda_{\max}(P)s^2, t)})/(\lambda_{\min}(P))$ with $\tilde{\beta} \in \mathcal{KL}$, and $\lambda(P) = (\lambda_{\min}(P))/(\lambda_{\max}(P))$.

For function $h(\mu_1) = -\mu_1^2 + \alpha\mu_1$, when $\mu_1 = (\alpha/2)$, it achieves the maximal value: $h_{\max} = (\alpha^2/4)$. Thus, let $\mu_1 = \alpha/2$, then, it follows from (16) that

$$\|e(t)\| \leq \max\left\{\tilde{\beta}(\|e_0\|, t-t_0), \left(\frac{4\|\omega\|_\infty^2}{\mu\alpha^2\lambda(P)}\right)^{\frac{1}{2}}\right\}. \tag{17}$$

Therefore, the estimation of $\|e(t)\|$ satisfies (17) with

$$\lim_{t\to\infty}\sup\|e(t)\| \leq \left(\frac{4\|\omega\|_\infty^2}{\mu\alpha^2\lambda(P)}\right)^{\frac{1}{2}}, \quad \mu > 1. \tag{18}$$

By letting $\mu \to 1^-$, we get that

$$\lim_{t\to\infty}\sup\|e(t)\| \leq \frac{2\|\omega\|_\infty}{\alpha\sqrt{\lambda(P)}}. \tag{19}$$

Hence, the actual synchronization scheme (1)–(3) achieves global uniform synchronization with an error bound $2/\alpha\sqrt{(\lambda_{\max}(P))/(\lambda_{\min}(P))}\|\omega\|_\infty$. The proof is thus complete. □

Remark 3.1: $(1')$ If there exist a positive definite matrix P and a positive constant $\alpha > 0$ such that condition (10) is replaced by:

$$e^TP(A_Ke + g(t,e,y)) \leq -\frac{\alpha}{2}e^TPe \tag{20}$$

then, the results of Theorem 3.1 still hold.

$(2')$ In the theoretical, for a given error bound $\varepsilon > 0$, by Theorem 3.1, if there exists a control gain matrix K and positive definite matrix P such that $\alpha \geq (2\|\omega\|_\infty)/(\varepsilon\sqrt{\lambda(P)})$ (for example, let $P = I$ and $K = -\mu H^{-1}$ with $\mu \geq \|A\| + L + (\|\omega\|_\infty)/(\varepsilon)$, then,

$$\|e(t)\| \leq \varepsilon, \quad \text{as} \quad t \longrightarrow \infty. \tag{21}$$

It should also be noticed that if ε is smaller, then it needs the control gain K (or μ) to be larger, which means the control cost will be higher. Hence, an optimal synchronization scheme can be synthesized into the following problem:

$$\min_{\|\omega\|_\infty<\infty}\int_{t_0}^{\infty}(e^T(t)Qe(t) + u^T(t)Ru(t))dt$$

where $Q > 0$ and $R > 0$ are some positive definite matrices. This is an interesting and meaningful problem.

In the following, considering the case $\varphi(t,x) = \varphi(x)$ and φ is a differentiable function, we investigate the error estimation by using nonlinear parametric variation technique.

Theorem 3.1:* Assume $\varphi(t,x) \equiv \varphi(x)$ and φ is differentiable and Assumptions 2.1–2.3 hold. And suppose that the control gain matrix $F = KH$ is chosen such that

$$\max_{x\in D}\left\{\lambda_{\max}\left(A + A^T + \varphi_x(x) + \varphi_x^T(x)\right)\right\} < \lambda_{\min}(F^T + F) \tag{22}$$

where $\varphi_x(x) = (\partial\varphi(x))/(\partial x)$ is the Jacobian matrix of function φ.

Then, the noise-free synchronization scheme (1)–(3) is exponentially synchronizing and the actual synchronization scheme (1)–(3) with noise is uniformly synchronizing with an error bound $(-\|\omega\|_\infty)/(\sigma_F)e^{(-L)/(\sigma_F)}$, where

$$\sigma_F \triangleq \frac{1}{2}\left\{\max_{x\in D}\left\{\lambda_{\max}\left(A + A^T + \varphi_x(x) + \varphi_x^T(x)\right)\right\} - \lambda_{\min}(F^T + F)\right\}.$$

Proof: Obviously, for any $x \in D$, $\varphi_x(x) + \varphi_x^T(x)$ is a symmetric matrix. Hence, there exists an orthogonal matrix $P(x)$ such that

$$\operatorname{diag}\{\lambda_1(x), \ldots, \lambda_n(x)\} = P(x)\left(\varphi_x(x) + \varphi_x^T(x)\right)P^T(x)$$

where $\lambda_i(x)$ is an eigenvalue of $\varphi_x(x) + \varphi_x^T(x)$, $i = 1, 2, \ldots, n$. Also, for $i = 1, 2, \ldots, n$, $\lambda_i(x)$ is a continuous function of the entries in matrix $\varphi_x(x) + \varphi_x^T(x)$. Hence, $\max_{1\leq i\leq n}\{\lambda_i(x)\}$ exists in the bounded and closed set D.

From the properties of a symmetric matrix, one has

$$\begin{aligned}\lambda_{\max}&\left\{(-F)^T + (-F) + A + A^T + \varphi_x(x) + \varphi_x^T(x)\right\}\\ &\leq \lambda_{\max}((-F)^T + (-F))\\ &\quad + \lambda_{\max}\left(A + A^T + \varphi_x(x) + \varphi_x^T(x)\right)\\ &\leq -\lambda_{\min}(F^T + F)\\ &\quad + \max_{x\in D}\left\{\lambda_{\max}\left(A + A^T + \varphi_x(x) + \varphi_x^T(x)\right)\right\} < 0.\end{aligned} \tag{23}$$

First, we show that the noise-free synchronization scheme (1)–(3) achieves exponentially synchronizing.

Let $\tilde{e}(t) = \tilde{e}(t, t_0, e_0)$ denote the solution of error system (4) under $\omega(t) = 0$. That is, $\tilde{e}(t)$ satisfies

$$\dot{\tilde{e}}(t) = A_F\tilde{e}(t) + g(t, \tilde{e}, y) \tag{24}$$

where $A_F = A - F = A - KH$.

Let $r(t) = r(t, t_0, r_0) = e(t) - \tilde{e}(t)$, then, $r(t)$ satisfies

$$\dot{r}(t) = A_Fr(t) + W(t) \tag{25}$$

where $W(t) = g(t,e,y) - g(t,\tilde{e},y) + \omega(t)$.

Clearly, by Assumption 2.3, $\|W(t)\| \leq L\|r(t)\| + \|\omega(t)\|$.

From (23), we get that all conditions of Theorem 2.1 in [46] are satisfied and thus it yields that

$$\|\tilde{e}(t)\| \leq \|x_0 - y_0\| e^{\sigma(F)(t-t_0)}. \tag{26}$$

This implies that the noise-free scheme (1)–(3) is exponentially synchronizing.

Second, we estimate the error $\|r(t)\|$.

Comparing the system (24) with its perturbed system (4), by Theorem 2.2 in [46], one has

$$\begin{aligned}\|r(t)\| &= \|e(t) - \tilde{e}(t)\| \\ &\leq \|r_0\| e^{\sigma_F(t-t_0)} + \int_{t_0}^{t} e^{\sigma_F(t-s)} \|W(s)\| ds \\ &\leq \|r_0\| e^{\sigma_F(t-t_0)} + \int_{t_0}^{t} e^{\sigma_F(t-s)} [L\|r(s)\| + \|\omega(s)\|] ds \\ &= \|r_0\| e^{\sigma_F(t-t_0)} + \frac{1}{\sigma_F}\left(e^{\sigma_F(t-t_0)} - 1\right)\|\omega\|_\infty \\ &\quad + L\int_{t_0}^{t} e^{\sigma_F(t-s)} \|r(s)\| ds. \end{aligned} \tag{27}$$

Since $\sigma_F < 0$, for any sufficiently small ε with $0 < \varepsilon \ll 1$, there exists a $T_1 \geq T_0$ such that for $t \geq T_1$

$$e^{\sigma_F(t-t_0)} < \varepsilon \ll 1. \tag{28}$$

By (27)–(28), for $t \geq T_1$, one has

$$\|r(t)\| \leq \left(\frac{-\|\omega\|_\infty}{\sigma_F} + \varepsilon\|r_0\|\right) + L\int_{t_0}^{t} e^{\sigma_F(t-s)} \|r(s)\| ds. \tag{29}$$

By the Gronwall-Bellman inequality, for $t \geq T_1$,

$$\begin{aligned}\|r(t)\| &\leq \left(\frac{-\|\omega\|_\infty}{\sigma_F} + \varepsilon\|r_0\|\right) e^{L\int_{t_0}^{t} e^{\sigma_F(t-s)} ds} \\ &\leq \left(\frac{-\|\omega\|_\infty}{\sigma_F} + \varepsilon\|r_0\|\right) e^{\frac{L}{\sigma_F}(e^{\sigma_F(t-t_0)} - 1)}. \end{aligned} \tag{30}$$

Finally, we estimate $\|e(t)\|$ and the synchronization error bound.

It follows from (30) that

$$\lim_{t\to\infty} \sup \|r(t)\| \leq \left(\frac{-\|\omega\|_\infty}{\sigma_F} + \varepsilon\|r_0\|\right) e^{\frac{-L}{\sigma_F}}. \tag{31}$$

Letting $\varepsilon \to 0$ yields

$$\lim_{t\to\infty} \sup \|r(t)\| \leq \frac{-\|\omega\|_\infty}{\sigma_F} e^{\frac{-L}{\sigma_F}}. \tag{32}$$

The exponential synchronization under free-noise implies that $\lim_{t\to\infty} \|\tilde{e}(t)\| = 0$. Hence, from (32), one has

$$\begin{aligned}\lim_{t\to\infty} \sup \|e(t)\| &\leq \lim_{t\to\infty} \sup(\|r(t)\| + \|\tilde{e}(t)\|) \\ &\leq \frac{-\|\omega\|_\infty}{\sigma_F} e^{\frac{-L}{\sigma_F}}. \end{aligned} \tag{33}$$

The proof is thus complete. □

Corollary 3.1:* If the controller matrix is designed as $K = \mu H^{-1}$, i.e., $F = \mu I$, where the constant μ satisfies

$$\mu > \frac{1}{2}\lambda_{\max}\left(A + A^T + \varphi_x(x) + \varphi_x^T(x)\right),\ x \in D$$

then the synchronization error has the following bound:

$$\|e(t)\| \leq \frac{\|\omega\|_\infty}{\mu - \beta} e^{\frac{L}{\mu - \beta}}$$

where $\beta = (1/2)\max_{z \in D}[\lambda_{\max}(A + A^T + \varphi_x(x) + \varphi_x^T(x))]$.

Proof: It is a direct consequence of Theorem 3.1*. □

Remark 3.1:* In Theorems 3.1–3.1*, two kinds of synchronization error estimation formulas have been proposed under which the uniform synchronization with an error bound can be achieved. The synchronization error estimation formula obtained in Theorem 3.1 is derived by using the Lyapunov function method, while the error estimation formulas in Theorem 3.1* and Corollary 3.1* are obtained by using the variation of parameters. The results in Theorem 3.1 may be relatively conservative due to many inequalities and Lemma 2.1.

B. Error Estimation and Uniform Synchronization for the Case With Time-Delays in TC

In this subsection, it is to study the synchronization error estimation problem in case of time-delays.

Case 1: There is no time-delay in slave system S.

If there is time-delay in TC but no time-delay in slave system S, then, the error system (5) be rewritten by

$$\dot{e}(t) = Ae(t) + g(t, e, y) + \hat{p}_\tau(t) + u_0(t) + \omega(t). \tag{34}$$

Theorem 3.2: Let $u = u_0 = -Kq(t)$, where K is the control gain matrix, and Assumption 2.1 and Assumption 2.3 hold and also assume that there exist a positive definite and symmetric matrix P and a positive constant $\alpha > 0$ such that (10) holds. Then, the noise-free synchronization scheme (1)–(3) achieves uniform synchronizing with an error bound

$$(2\tau(\|A\| + L))/(\alpha)\sqrt{(\|(KH)^T P(KH)\|)/(\lambda_{\min}(P))}\|x\|_\infty$$

and the actual synchronization scheme (1)–(3) with noise achieves global uniform synchronization with an error bound shown at the top of the next page.

Proof: Let $V(e) = e^T Pe$ and $e(t, t_0, e_0)$ denote the solution of (34) passing through the initial point (t_0, ϕ). Denote $\tilde{K} = KH$. By (10) and similar Proof of Theorem 3.1, one has

$$\begin{aligned} D^+V(e)|_{(34)} &= 2e^T(A_K e + g(t, e, y)) \\ &\quad + 2e^T PKH(x(t-\tau) - x(t)) + 2e^T P\omega(t) \\ &\leq -\alpha V(e) + \mu_1 V(e) + \mu_1^{-1}\|P\|\|\omega(t)\|^2 \\ &\quad + \mu_2 V(e) + \mu_2^{-1}\|\tilde{K}^T P\tilde{K}\|\|x(t-\tau) - x(t)\|^2 \\ &= -(\alpha - \mu_1 - \mu_2)V(e) + \mu_1^{-1}\|P\|\|\omega\|_\infty^2 \\ &\quad + \mu_2^{-1}\|\tilde{K}^T P\tilde{K}\|\|x(t-\tau) - x(t)\|^2 \end{aligned} \tag{35}$$

$$(3/\alpha)\sqrt{(3[\|P\|\|\omega\|_\infty^2+\tau^2\|(KH)^TPKH\|(\|A\|+L)^2\|x\|_\infty^2])/(\lambda_{\min}(P))}$$

where μ_1, μ_2 are some positive constants satisfying $\mu_1 > 0, \mu_2 > 0$, and $\mu_1 + \mu_2 < \alpha$.

Since φ is Lipschitz, we have

$$\|x(t-\tau)-x(t)\| = \left\|\int_{t-\tau(t)}^{t}(Ax(s)+\varphi(s,x(s)))ds\right\| \le \tau(\|A\|+L)\|x_\tau(t)\| \le \tau(\|A\|+L)\|x\|_\infty \quad (36)$$

where $\|x_\tau(t)\| = \max_{t-\tau\le s\le t}\{\|x(s)\|\}$.

Thus, if there exists no noise in the TC, i.e., $\omega(t)\equiv 0$, then by (35)–(36) and similar proof of (17) in Theorem 3.1, we have

$$\|e(t)\| \le \tilde{\beta}(\|\phi\|_\tau, t-t_0) + \frac{2\tau(\|A\|+L)}{\alpha}\sqrt{\frac{\|\tilde{K}^TP\tilde{K}\|}{\lambda_{\min}(P)}}\|x\|_\infty \quad (37)$$

where $\tilde{\beta}\in\mathcal{KL}$, which implies that

$$\lim_{t\to\infty}\sup\|e(t)\| \le \frac{2\tau(\|A\|+L)}{\alpha}\sqrt{\frac{\|\tilde{K}^TP\tilde{K}\|}{\lambda_{\min}(P)}}\|x\|_\infty. \quad (38)$$

Hence, the actual synchronization scheme (1)–(3) achieves global uniform synchronization with an error bound $(2\tau(\|A\|+L))/(\alpha)\sqrt{(\|\tilde{K}^TP\tilde{K}\|)/(\lambda_{\min}(P))}\|x\|_\infty$.

On the other hand, if there exists noise $\omega(t)\ne 0$ in the TC, then, from (35)–(36), we get

$$D^+V(e)\,|_{(34)} \le -(\alpha-\mu_1-\mu_2)V(e)+\mu_1^{-1}\|P\|\|\omega\|_\infty^2 + \tau^2\mu_2^{-1}\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2. \quad (39)$$

Thus, when

$$V(e) \ge \frac{\mu_1^{-1}\|P\|\|\omega\|_\infty^2+\tau^2\mu_2^{-1}\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2}{\mu(\alpha-\mu_1-\mu_2)}$$

where $\mu < 1$, then

$$D^+V(e)\,|_{(34)} \le -(1-\mu)(\alpha-\mu_1-\mu_2)V(e). \quad (40)$$

By Lemma 2.1, we get that there exists a function $\beta\in\mathcal{KL}$ with $\beta(s,0)\ge s$ such that

$$V(e) \le \max\{\beta(V(\phi), t-t_0), \nu\} \quad (41)$$

where

$$\nu = \frac{\mu_1^{-1}\|P\|\|\omega\|_\infty^2+\tau^2\mu_2^{-1}\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2}{\mu(\alpha-\mu_1-\mu_2)}.$$

For function $v(\mu_1,\mu_2)=\mu_1\mu_2(\alpha-\mu_1-\mu_2)$, when $\mu_1=\mu_2=(\alpha/3)$, it achieves the maximal value: $v_{\max}=(\alpha^3)/(27)$. Thus, it follows from (41) that

$$V(e) \le \max\{\beta(V(\phi), t-t_0), \tilde{\nu}\} \quad (42)$$

where

$$\tilde{\nu} = \frac{27\left[\|P\|\|\omega\|_\infty^2+\tau^2\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2\right]}{\mu\alpha^2}.$$

Thus, we have

$$\|e(t)\| \le \tilde{\beta}(\|\phi\|_\tau, t-t_0)+\sqrt{\frac{\tilde{\nu}}{\lambda_{\min}(P)}} \quad (43)$$

where

$$\tilde{\beta}(s,t) = (\sqrt{\beta(\lambda_{\max}(P)s^2, t)})/(\lambda_{\min}(P))$$

with $\tilde{\beta}\in\mathcal{KL}$.

Therefore, letting $\mu\to 1^-$, we obtain

$$\lim_{t\to\infty}\sup\|e(t)\| \le \frac{1}{\alpha}\sqrt{\frac{27\left[\|P\|\|\omega\|_\infty^2+\tau^2\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2\right]}{\lambda_{\min}(P)}}. \quad (44)$$

Hence, the actual synchronization scheme (1)–(3) achieves global uniform synchronization with an error bound shown at the bottom of the page. The proof is thus complete. □

Case 2: There is time-delay τ in slave system S.

In this part, we investigate the case that there is time-delay τ in TC and in slave system S by using the ISS results established for nonlinear systems [24]–[29]. We consider the following problem:

Problem formulation: If we have designed some control gain matrix K such that the noise-free and time-delay-free synchronization scheme (1)–(3) achieves global exponential synchronization, does there exist a positive constant $\tau^* > 0$ such that for any $\tau\in[0,\tau^*)$, the synchronization scheme (1)–(3) with

$$(1/\alpha)\sqrt{(27[\|P\|\|\omega\|_\infty^2+\tau^2\|\tilde{K}^TP\tilde{K}\|(\|A\|+L)^2\|x\|_\infty^2])/(\lambda_{\min}(P))}$$

noise and time-delays in TC achieves global uniform synchronization with an estimated error bound under the control input $u = u_\tau = -Kq_\tau(t)$?

If there is time-delay τ in TC and in slave system S, then, the error system (5) be rewritten by

$$\dot{e}(t) = Ae(t) + g(t,e,y) + \hat{p}_\tau(t) + u_\tau(t) + \omega(t). \tag{45}$$

Theorem 3.3: Let $u = u_\tau = -Kq_\tau(t)$ and Assumption 2.1 and Assumption 2.3 hold, and also assume that there exist a positive definite and symmetric matrix P and a positive constant $\alpha > 0$ satisfying

$$\alpha > 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\tau\|KH\|(\|A\| + L + \|KH\|) \tag{46}$$

such that, for any solution $e(t)$ of (45), inequality (10) holds. Then, the noise-free synchronization scheme (1)–(3) achieves global exponential synchronization and the actual synchronization scheme (1)–(3) with noise and time-delays in TC achieves global uniform synchronization with an error bound $(\lambda_{\max}(P))/(\lambda_{\min}(P)) \cdot (\tau\|KH\| + 1)/((\alpha/2) - \tau\|KH\|(\|A\| + L + \|KH\|))\|\omega_\tau\|_\infty$.

Proof: Let Lyapunov function be $V(e) = e^TPe$ and $e(t) = e(t,t_0,\phi)$ denote the solution of (45) satisfying the initial condition (t_0,ϕ). Denote $z(t) = KH(e(t) - e(t-\tau(t)))$, and without loss of generality, denote $\|e_\tau(t)\| = \max_{t-2\tau\le s\le t}\|e(s)\|$. Since φ is Lipschitz, we have

$$\begin{aligned}\|z(t)\| &\le \|KH\|\|e(t) - e(t-\tau(t))\|\\ &= \|KH\|\left\|\int_{t-\tau(t)}^{t}(Ae(s) + g(s,e(s),y(s))\right.\\ &\quad - KHe(s-\tau(s)) + \omega(s))ds\|\\ &\le \|KH\|\int_{t-\tau}^{t}\|Ae(s) + g(s,e(s),y(s))\\ &\quad - KHe(s-\tau(s)) + \omega(s)\|ds\\ &\le \tau[\|KH\|(\|A\| + L + \|KH\|)\|e_\tau(t)\|\\ &\quad + \|KH\|\|\omega_\tau(t)\|].\end{aligned} \tag{47}$$

It follows from (45) and (47) that

$$\begin{aligned}&D^+V(e)|_{(45)}\\ &\quad= 2e^TP[Ae + g(t,e,y) + \hat{p}(t) + u_\tau + \omega]\\ &\quad= 2e^TP[A_Ke + g(t,e,y)] + 2e^TPz(t) + 2e^TP\omega(t)\\ &\quad\le -\alpha e^TPe + 2\|e(t)\|\|P\|\|z(t)\| + 2\|e(t)\|\|P\|\|\omega(t)\|\\ &\quad\le -\alpha e^TPe + 2\|e(t)\|\|P\|\\ &\qquad\times[\tau(\|KH\|(\|A\| + L + \|KH\|)\\ &\qquad\cdot\|e_\tau(t)\| + \|KH\|\|\omega_\tau(t)\|) + \|\omega(t)\|]\\ &\quad\le -\alpha V(e) + 2\|e(t)\|\|P\|\\ &\qquad\times[\tau\|KH\|(\|A\| + L + \|KH\|)\\ &\qquad\cdot\|e_\tau(t)\| + (\tau\|KH\| + 1)\|\omega_\tau(t)\|]\\ &\quad\le -\alpha V(e) + \sigma V_\tau(e(t))\\ &\qquad + 2\|e(t)\|\|P\|(\tau\|KH\| + 1)\|\omega_\tau(t)\|\end{aligned} \tag{48}$$

where $V_\tau(e(t)) = \max_{t-2\tau\le s\le t}\{e^T(s)Pe(s)\}$, and $\sigma = 2(\lambda_{\max}(P))/(\lambda_{\min}(P))\tau\|KH\|(\|A\| + L + \|KH\|)$.

Since $\alpha > \sigma$, there exists a positive number $q > 1$ such that $\alpha > q\sigma$. Hence, if there exists no noise in the TC, i.e., $\omega(t) \equiv 0$, then by (48), if $qV(e(t)) \ge V_\tau(e(t))$, we have

$$D^+V(e) \le -(\alpha - q\sigma)V(e). \tag{49}$$

It follows from Lemma 2.2 that

$$\|e(t)\| \le \|\phi\|_\tau e^{-\frac{\alpha_1}{2}(t-t_0)} \tag{50}$$

where $\alpha_1 = \min\{\alpha - q\sigma, (\ln q)/(\tau)\} > 0$.

Hence, the noise-free synchronization scheme (1)–(3) achieves global exponential synchronization.

On the other hand, if $\omega(t) \ne 0$, define two functions $\gamma_1, \gamma_2 \in \mathcal{K}$ as: for any $s \in R_+$

$$\begin{aligned}\gamma_1(s) &= \frac{\tau\|KH\|s}{\mu_1}\\ \gamma_2(s) &= \frac{(\tau\|KH\| + 1)s}{\mu_2\left(\frac{\alpha}{2} - \mu_1(\|A\| + L + \|KH\|)\right)}\end{aligned} \tag{51}$$

where μ_1, μ_2 satisfy

$$\tau\|KH\| < \mu_1 < \frac{\lambda(P) - \mu_2}{1 - \mu_2}\cdot\frac{\alpha}{2(\|A\| + L + \|KH\|)},\quad 0 < \mu_2 < \lambda(P) \le 1 \tag{52}$$

where $\lambda(P) = (\lambda_{\min}(P))/(\lambda_{\max}(P))$.

If $\|e(t)\| \ge \max\{\gamma_1(\|e_\tau(t)\|), \gamma_2(\|\omega_\tau(t)\|)\}$, then it follows from (48) and (51)–(52) that

$$\begin{aligned}D^+V(e) &\le -2(1-\mu_2)\left[\frac{\lambda(P) - \mu_2}{1-\mu_2}\cdot\frac{\alpha}{2}\right.\\ &\qquad\left. - \mu_1(\|A\| + L + \|KH\|)\right]\|e\|^2\|P\|\\ &= -a(V(e))\end{aligned} \tag{53}$$

where $a(s) = 2(1-\mu_2)[(\lambda(P) - \mu_2)/(1-\mu_2)\cdot(\alpha/2) - \mu_1(\|A\| + L + \|KH\|)]s$ for all $s \in R_+$.

From (52)–(53), we get that $a \in \mathcal{K}$ and $\gamma_1(s) < s$ for all $s \in R_+$. Hence, by Theorem 2 in [26], we get that the system (45) is ISS with gain γ_2 and hence there exists a function $\beta \in \mathcal{KL}$ with $\beta(s,0) \ge s$ such that

$$\begin{aligned}\|e(t)\| &\le \max\left\{\beta(\|\phi\|, t-t_0),\right.\\ &\qquad\left.\frac{\tau\|KH\| + 1}{\mu_2\left(\frac{\alpha}{2} - \mu_1(\|A\| + L + \|KH\|)\right)}\|\omega_\tau\|_\infty\right\}\\ &\le \beta(\|\phi\|, t-t_0)\\ &\qquad + \frac{\tau\|KH\| + 1}{\mu_2\left(\frac{\alpha}{2} - \mu_1(\|A\| + L + \|KH\|)\right)}\|\omega_\tau\|_\infty\end{aligned} \tag{54}$$

with

$$\begin{aligned}&\lim_{t\to\infty}\sup\|e(t)\|\\ &\qquad\le \frac{\tau\|KH\| + 1}{\mu_2\left(\frac{\alpha}{2} - \mu_1(\|A\| + L + \|KH\|)\right)}\|\omega_\tau\|_\infty\end{aligned} \tag{55}$$

where μ_1, μ_2 satisfying (52).

By letting $\mu_1 \longrightarrow (\tau\|KH\|)^+$ and $\mu_2 \longrightarrow \lambda(P)^-$, we get that

$$\lim_{t\to\infty} \sup \|e(t)\| \le \frac{\tau\|KH\|+1}{\lambda(P)\left(\frac{\alpha}{2} - \tau\|KH\|(\|A\|+L+\|KH\|)\right)}\|\omega_\tau\|_\infty. \quad (56)$$

Then, the actual synchronization scheme (1)–(3) achieves global uniform synchronization with an error bound $(\lambda_{\max}(P))/(\lambda_{\min}(P)) \cdot (\tau\|KH\| + 1)/((\alpha/2) - \tau\|KH\|(\|A\|+L+\|KH\|))\|\omega_\tau\|_\infty$. The proof is complete.□

Remark 3.2: (1′) Let

$$\tau^* = \frac{\alpha}{2\|KH\|(\|A\|+L+\|KH\|)}. \quad (57)$$

Then, by Theorem 3.3, τ^* is a upper bound for the time-delay such that for any $0 \le \tau < \tau^*$ the actual synchronization scheme (1)–(3) provides global uniform synchronization with an error bound $(\lambda_{\max}(P))/(\lambda_{\min}(P))\cdot(\tau\|KH\|+1)/((\alpha/2)-\tau\|KH\|(\|A\|+L+\|KH\|))\|\omega_\tau\|_\infty$.

(2′) In Theorem 3.3, if $\tau(t) \equiv 0$ for all $t \in R_+$, i.e., $\tau = 0$, then, the results of Theorem 3.1 can be derived from Theorem 3.3. Hence, Theorem 3.3 is the generalization of Theorem 3.1.

Corollary 3.1: Let $K = -\mu H^{-1}$. Let Assumptions 2.1 and 2.3 hold, then, there is a maximal upper bound $\tau^*_{\max}$ of time-delay:

$$\tau^*_{\max} = \frac{(\sqrt{2}-1)^2}{\|A\|+L}. \quad (58)$$

such that, for any $0 \le \tau < \tau^*_{\max}$, and for any control gain matrix $K = -\mu H^{-1}$ with $\mu > \|A\| + L$, the actual synchronization scheme (1)–(3) achieves global uniform synchronization with an error bound $((\tau\mu+1)\|\omega_\tau\|_\infty)/(\mu(1-\tau(\mu+\|A\|+L)) - \|A\| - L)$.

Moreover, for any fixed τ satisfying $0 < \tau < \tau^*_{\max}$, the control gain matrix $K = -\mu^* H^{-1}$ with $\mu^* = (\sqrt{2}-1)/(\tau)$ will achieve global uniform synchronization with an minimal error bound $(\tau(\sqrt{2}+1)\|\omega_\tau\|_\infty)/((\sqrt{2}-1)-\tau(\|A\|+L)(\sqrt{2}+1))$.

Proof: Since $u = u_\tau = -\mu e_\tau$, then, $u_0 = -\mu e$ and hence by Assumption 2.3, we get

$$\begin{aligned} &e^T(Ae + g(t,e,y) + u_0)\\ &\quad = e^T(Ae + f(t,y) - f(t,y-e) - \mu e)\\ &\quad \le |e^T(Ae + f(t,y) - f(t,y-e))| - \mu e^T e\\ &\quad \le -(\mu - \|A\| - L)e^T e \end{aligned} \quad (59)$$

which implies that $\alpha = 2(\mu - \|A\| - L)$.

Hence, it follows from Theorem 3.3 and Remark 3.2 that

$$\tau^* = \frac{\alpha}{\|KH\|(\|A\|+L+\|KH\|)} = \frac{\mu - \|A\| - L}{\mu(\mu+\|A\|+L)} \quad (60)$$

is the upper bound of time-delay such that for any $0 \le \tau < \tau^*$ the actual synchronization scheme (1)–(3) is global uniform synchronization with an error bound $((\tau\mu+1))/(\mu(1-\tau(\mu+\|A\|+L)) - \|A\| - L)\|\omega_\tau\|_\infty$.

Note for $\mu > \|A\| + L > 0$, the function

$$\varphi(\mu) = \frac{\mu - \|A\| - L}{\mu(\mu+\|A\|+L)}$$

reaches its maximal value at $\mu^{**} = (1+\sqrt{2})(\|A\|+L)$ and it is strictly decreasing for $\mu > \mu^{**}$ and $\lim_{\mu\to\infty}\varphi(\mu) = 0$. Therefore, by setting $\mu = \mu^{**}$, then $\tau^* = \tau^*_{\max} = (\mu^{**} - \|A\| - L)/(\mu^{**}(\mu^{**}+\|A\|+L)) = ((\sqrt{2}-1)^2)/(\|A\|+L)$. Hence, for any time-delay τ of signal in TC satisfying $\tau < \tau^*_{\max} = ((\sqrt{2}-1)^2)/(\|A\|+L)$ and for any $\mu > \|A\| + L$, the linear controller $u = u_\tau = -\mu e_\tau$ achieves global uniform synchronization with an error bound $((\tau\mu+1))/(\mu(1-\tau(\mu+\|A\|+L)) - \|A\| - L)\|\omega_\tau\|_\infty$.

Moreover, if $0 < \tau < \tau^*_{\max}$, then, the function $\tilde{\varphi}(\mu) = ((\tau\mu+1))/(\mu(1-\tau(\mu+\|A\|+L)) - \|A\| - L)$ reaches its minimal value $\tilde{\varphi}_{\min}$ at $\mu^* = (\sqrt{2}-1)/(\tau)$, where $\tilde{\varphi}_{\min} = \tilde{\varphi}(\mu^*) = (\tau(\sqrt{2}+1)\|\omega_\tau\|_\infty)/((\sqrt{2}-1)-\tau(\|A\|+L)(\sqrt{2}+1))$. The proof is complete. □

Remark 3.3: Note that the maximal upper bound $\tau^*_{\max}$ of time-delay is independent of linear control input $K = -\mu H^{-1}$. It is only determined by the property of the system used for synchronization scheme. Meantime, Corollary 3.1 can be used to design a linear control input so that the synchronization error achieves the minimal value.

Remark 3.4: In the literature, if the stability issue is investigated for time-delayed systems, the Lyapunov–Krasovskii function is often used to derive the delay-dependent stability criteria. Recently, in [34]–[37], delay-dependent *asymptotical synchronization* criteria for Lur'e systems are established by using this kind of Lyapunov function. The Lyapunov–Krasovskii function is often in form of

$$V(e) = e^T Pe + \int_{t-\tau}^{t}\int_{t+\theta}^{t} e^T(s)Qe(s)ds + \int_{t-\tau}^{t}\int_{t+\theta-\tau}^{t} e^T(s)Ze(s)ds \quad (61)$$

where P is positive definite matrix and Q, Z are nonnegative definite matrices. In the following, in order to make some comparison between Lyapunov–Krasovskii function approach and Razumikhin technique, we employ Lyapunov–Krasovskii function method to analyze the same synchronization issue. Similar to [36], we take a Lyapunov–Krasovskii function as

$$V(e) = V_1(e) + V_2(e) + V_3(e) + V_4(e) \quad (62)$$

where for some positive constants $r_1 > 0, r_2 > 0, r_3 > 0$,

$$\begin{aligned} V_1(e) &= e^T Pe\\ V_2(e) &= r_1\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)A^T Ae(s)ds\\ V_3(e) &= r_2\int_{-\tau}^{0}\int_{t+\theta-\tau}^{t} e^T(s)F^T Fe(s)ds\\ V_4(e) &= r_3\int_{-\tau}^{0}\int_{t+\theta}^{t} g(s,e(s),y(s))^T\\ &\qquad \times g(s,e(s),y(s))ds. \end{aligned}$$

Rewrite the error system (45) as

$$\dot{e}(t) = A_F e(t) + \tilde{g}(t) + \omega(t) - F\int_{-\tau}^{0}\dot{e}(t+\theta)d\theta \quad (63)$$

where $A_F = A + F, F = KH, \tilde{g}(t) = g(t, e, y)$. Then, by taking the time derivative along the error system (63) and using inequalities in [45], for some positive constants $r_4 > 0, r_5 > 0$, we obtain

$$D^+V(e) \le e^T\Phi e + (r_4\tau + r_5\|P\|\|)\|\omega_\tau\|_\infty^2 \tag{64}$$

where

$$\Phi = PA_F + A_F^TP + 2L\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}P + r_1\tau A^TA + r_2\tau F^TF + \left(r_3\tau L^2 + \frac{1}{r_5}\|P\|\right)I + \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)\tau PFF^TP.$$

Thus, if the TC is noise-free and Φ is a negative definite matrix for some $P > 0$ and constants $r_i > 0, (i = 1, 2, 3, 4, 5)$, i.e., $\omega(t) \equiv 0$ and $\Phi < 0$, then, the error system (63) is asymptotically stable and hence *asymptotical synchronization* under the control scheme can be achieved. Moreover, by using $\Phi < 0$ and LMI technique, the maximal time-delay τ^* can be estimated. For example, if we take $P = I$, $F = -\mu I$, i.e., $K = -\mu H^{-1}$, and $r_1 = r_2 = r_3 = r_4 = r_5 = 1$, then, we have $\Phi = (A + A^T + \tau A^TA) + (5\tau\mu^2 - 2\mu + \tau L^2 + 2L + 1)I$. From $\Phi < 0$ and by using the similar analysis in Corollary 3.1, we can estimate the control gain μ^* and the maximal time-delay τ^*.

However, it should be noted that there are two aspects of difficulties in using Lyapunov–Krasovskii function to analyze ISS or synchronization issue of time-delays systems under the external disturbances. The first one is: under the TC noise $\omega(t) \ne 0$, as it can be seen from (64), even $\Phi < 0$, it is hard to derive the error estimation formula of $e(t)$. The reason is that the Lyapunov–Krasovskii function often has complex structure (it is a sum of a quadric positive definite function and several nonnegative functions with integration) and thus it is hard to use Lemma 2.1 to derive ISS properties of error system (63). The second one lies in: most results obtained in the literature by using Lyapunov–Krasovskii function are asymptotically stable, not exponentially stable. Thus, even for synchronization scheme under noise-free TC, i.e., $\omega(t) \equiv 0$, it is also hard to derive the exponential synchronization criterion and the synchronization speed expressed by Lyapunov exponent. But from Theorem 3.3 and Corollary 3.1, by making use of the Razumikhin type Lemma 2.2 and combining the ISS type Lemma 2.1, the error estimation formula of $e(t)$ and exponential synchronization criteria and the Lyapunov exponents are all obtained.

Case 3: Extension to the general case: there is time-delay $\tau_1 (0 \le \tau_1 \le \tau)$ in slave system S.

If there is time-delay τ in TC and time-delay τ_1 $(0 \le \tau_1 \le \tau)$ in slave system S, then, the error system (5) be rewritten by:

$$\dot{e}(t) = Ae(t) + g(t, e, y) + \hat{p}_\tau(t) + u_{\tau_1}(t) + \omega(t). \tag{65}$$

Theorem 3.4: Let $u = u_{\tau_1} = -Kq_{\tau_1}(t)$ and Assumption 2.1 and Assumption 2.3 hold, and also assume that there exist a positive definite and symmetric matrix P and a positive constant $\alpha > 0$ satisfying (46) such that, for any solution $e(t)$ of (65), inequality (10) holds. Then, the noise-free synchronization scheme (1)–(3) achieves global synchronization with an error bound $(\lambda_{\max}(P))/(\lambda_{\min}(P)) \cdot ((\tau - \tau_1)\|KH\|(\|A\| + L)(\tau_1\|KH\| + 1)\|x\|_\infty)/((\alpha/2) - \tau_1\|KH\|(\|A\| + L + \|KH\|))$, and the actual synchronization scheme (1)–(3) with noise and time-delays in TC achieves global uniform synchronization with an error bound $(\lambda_{\max}(P))/(\lambda_{\min}(P)) \cdot ((\tau - \tau_1)\|KH\|(\|A\| + L)(\tau_1\|KH\| + 1)\|x\|_\infty + (\tau_1\|KH\| + 1)\|\omega_\tau\|_\infty)/(\alpha/2 - \tau_1\|KH\|(\|A\| + L + \|KH\|))$.

Proof: Let Lyapunov function be $V(e) = e^TPe$ and $e(t) = e(t, t_0, \phi)$ denote the solution of (65) satisfying the initial condition (t_0, ϕ). Denote $z(t) = KH(e(t) - e(t - \tau_1))$, and without loss of generality, denote $\|e_\tau(t)\| = \max_{t-2\tau \le s \le t}\|e(s)\|$. Since φ is Lipschitz, we have

$$\begin{aligned}\|x(t-\tau) - x(t-\tau_1)\| &\le \left\|\int_{t-\tau}^{t-\tau_1}(Ax(s) + \varphi(s, x))ds\right\| \\ &\le (\tau - \tau_1)(\|A\| + L)\|x_\tau(t)\|\end{aligned} \tag{66}$$

and

$$\begin{aligned}\|e(t) - e(t-\tau_1)\| &= \left\|\int_{t-\tau_1}^{t}(Ae(s) + g(s, e(s), y(s)) + \omega(s) - KHe(s-\tau_1) + KH(x(t-\tau) - x(t-\tau_1)))ds\right\| \\ &\le \tau_1\left[(\|A\| + L + \|KH\|)\|e_\tau(t)\| + \|\omega_\tau(t)\| + (\tau - \tau_1)\|KH\|(\|A\| + L)\|x_\tau(t)\|\right].\end{aligned} \tag{67}$$

It follows from (65) and (66)–(67) that

$$\begin{aligned}D^+V(e)|_{(65)} &= 2e^TP[Ae + g(t, e, y) + \hat{p}_\tau(t) + u_\tau + \omega] \\ &= 2e^TP[A_Ke + g] + 2e^TPz(t) + 2e^TP\omega(t) + 2e^TPKH(x(t-\tau) - x(t-\tau_1)) \\ &\le -\alpha e^TPe + 2\|e(t)\|\|P\|\|z(t)\| + 2\|e(t)\|\|P\|\|\omega(t)\| + 2\|e(t)\|\|P\|\|KH\|\|x(t-\tau) - x(t-\tau_1)\| \\ &\le -\alpha V(e) + \sigma V_\tau(e(t)) + 2\theta\|e\|\|P\|\|x_\tau(t)\| + 2\|e(t)\|\|P\|(\tau_1\|KH\| + 1)\|\omega_\tau(t)\|\end{aligned} \tag{68}$$

where $V_\tau(e(t)) = \max_{t-2\tau \le s \le t}\{e^T(s)Pe(s)\}$, $\sigma = 2(\lambda_{\max}(P))/(\lambda_{\min}(P))\tau_1\|KH\|(\|A\| + L + \|KH\|)$, and $\theta = \|KH\|(\tau - \tau_1)(\|A\| + L)(\tau_1\|KH\| + 1)$.

By the similar proofs as in Theorem 3.2 and Theorem 3.3, we can derive all the results. The details are omitted here. The proof is complete. □

IV. Examples

In this section, two representative examples are given for illustration. Here, the numerical simulation procedure is coded and executed by using the Runge-Kutta integration rule with adaptive step size (ode23 in Matlab).

Example 4.1: Consider the chaotic Chua's circuit [5]

$$\begin{cases}\dot{x_1} &= \alpha(x_2 - h(x_1)) \\ \dot{x_2} &= x_1 - x_2 + x_3 \\ \dot{x_3} &= -\beta x_2\end{cases} \tag{69}$$

where $h(x_1) = m_1x_1 + (1/2)(m_0 - m_1)(|x_1 + c| - |x_1 - c|)$.

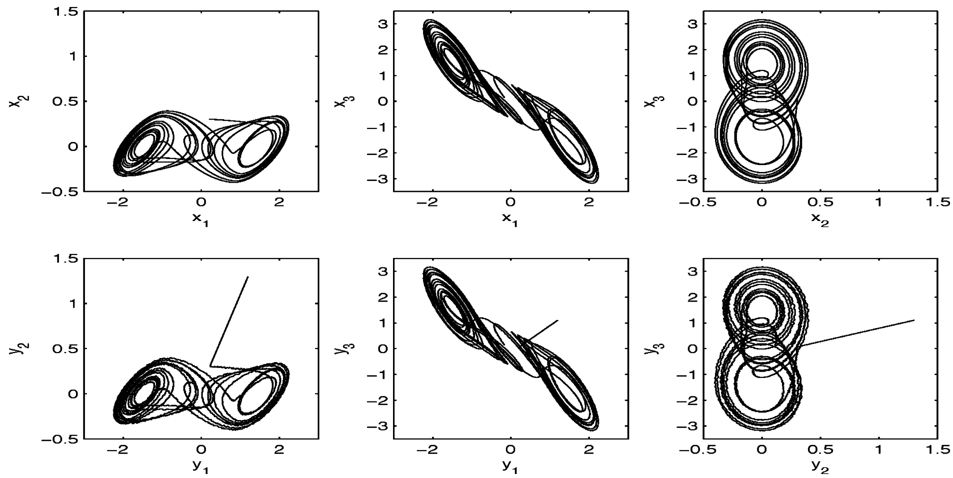
Fig. 2. Attractors of master system (up) and slave system (down).

It is well known that with parameters $m_0 = -1/7, m_1 = 2/7, \alpha = 9, \beta = 14.28, c = 1$, for which the sector condition is $[0, 1]$, i.e., with slope $\kappa = 1$, the circuit generates a double-scroll attractor.

Chua's circuit can be represented in a general Lur'e form: $\dot{\xi} = A\xi + B\sigma(C\xi)$, with $\xi = (\,x_1 \quad x_2 \quad x_3\,)^T$, and

$$A = \begin{pmatrix} -\alpha m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -\alpha(m_0 - m_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma(C\xi) = (\,\tfrac{1}{2}(|x_1 + c| - |x_1 - c|) \quad 0 \quad 0\,)^T.$$

For any $\xi, \eta \in D$, one has

$$\begin{aligned} &\|f(\xi) - f(\eta)\| \\ &\quad = \|A(\xi - \eta) + B(\sigma(C\xi) - \sigma(C\eta))\| \\ &\quad \le (\|A\| + \kappa\|B\|\|C\|)\|\xi - \eta\|. \end{aligned} \tag{70}$$

Thus, $\|A\| + L = \|A\| + \kappa\|B\|\|C\| = 20.8277$.

Set $u = u_0 = KHe, H = I$, and $K = -\mu I, \mu > 0$, then

$$\begin{aligned} &e^T(Ae + g + KHe) \\ &\quad = \frac{1}{2}e^T(A + A^T + K^T + K)e \\ &\qquad + e^T B[\sigma(C(e + y)) - \sigma(Cy)] \\ &\quad \le \frac{1}{2}e^T(A + A^T - 2\mu I)e + \kappa\|B\|\|C\|e^T e \\ &\quad = (12.1727 - \mu)e^T e \triangleq -\alpha e^T e. \end{aligned} \tag{71}$$

Hence, if there is no time-delay in TC and slave system S, then, by Theorem 3.1, for sufficiently large t, one obtains an estimation of the synchronization error as

$$\|e(t)\| \le \frac{2\|\omega\|_\infty}{\alpha} = \frac{2\|\omega\|_\infty}{\mu - 12.1727}. \tag{72}$$

Therefore, by setting $\mu \gg 12.1727$ be sufficiently large, the synchronization error will be sufficiently small and less than $\|\omega\|_\infty$.

In simulations, initial values are $[x_1\ x_2\ x_3\ y_1\ y_2\ y_3] = [0.2\ 0.3\ 0.1\ 1.2\ 1.3\ 1.1]$. Let $\mu = 20$, then, for sufficiently large $t, \|e(t)\| \le 0.1278\|\omega\|_\infty$. Some noise is added, in which $\|\omega\|_\infty = 0.1$. The noise is uniformly distributed on $[-\|\omega\|_\infty, \|\omega\|_\infty]$, generated by using $(\mathrm{rand}(1) - 0.5) \times 2 \times \|\omega\|_\infty$ in Matlab. The attractors of the master and slave systems and the errors of synchronization are shown in Figs. 2–3, respectively, from which one can observe that all the synchronization errors are less than 0.015, consistent with the results of Theorem 3.1.

Example 4.2: Consider Colpitts' oscillator [10]

$$\begin{cases} \dot{x_1} = \alpha x_2 \\ \dot{x_2} = -\sigma(x_1 + \gamma x_2 + x_3) \\ \dot{x_3} = \beta(x_2 + a_1 x_1 + a_3 x_1^3) \end{cases} \tag{73}$$

where $\alpha, \beta, \sigma, a_1, \gamma, a_3 \in R$. With parameters $\alpha = 2.4,\ \beta = 2.2,\ \sigma = 1,\ \gamma = 0.252,\ a_1 = 1$, and $a_3 = -0.2$, the system is chaotic.

Letting $X = (x_1, x_2, x_3)^T$ convert Colpitts' oscillator to

$$\dot{X} = AX + \Phi(X) \tag{74}$$

where

$$A = \begin{pmatrix} 0 & \alpha & 0 \\ -\sigma & -\gamma\sigma & -\sigma \\ a_1\beta & \beta & 0 \end{pmatrix} \text{ and } \Phi(X) = \begin{pmatrix} 0 \\ 0 \\ a_3\beta x_1^3 \end{pmatrix}.$$

For any $X, Y \in D$, one has

$$\begin{aligned} \|f(X) - f(Y)\| &= \|A(X - Y) + \Phi(X) - \Phi(Y)\| \\ &\le (\|A\| + 6\delta^2|a_3\beta|)\|X - Y\| \end{aligned} \tag{75}$$

where $\|X\|, \|Y\| \le \delta$. Hence, $\|A\| + L = \|A\| + 6\delta^2|a_3\beta| = 3.7322 + 2.64\delta^2$, and Note that, $\lambda_{\max}\{A + A^T\} = 3.1308$. Set $u = u_\tau = KHe_\tau$, and $H = I, K = -\mu I, \mu > 0$, then

$$\begin{aligned} e^T(Ae + g + u) =& \frac{1}{2}e^T(A + A^T + K^T + K)e \\ &+ e^T[\Phi(Y) - \Phi(Y - e)] \end{aligned}$$

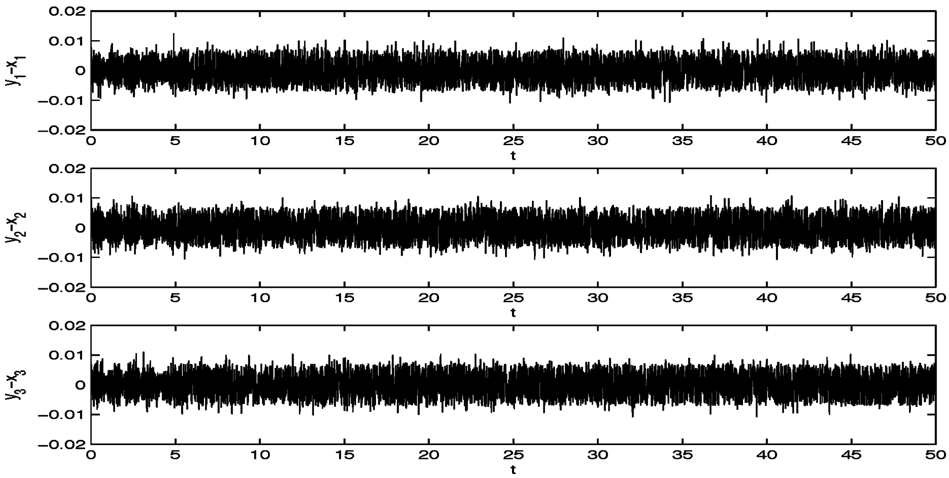

Fig. 3. Synchronization errors with $\|\omega\|_\infty = 0.1$.

$$\le \frac{1}{2}e^T(A + A^T - 2\mu I)e + 6\delta^2|a_3\beta|e^T e$$
$$= \frac{1}{2}(3.1308 + 5.28\delta^2 - 2\mu)e^T e \triangleq -\frac{\alpha}{2}e^T e. \quad (76)$$

Hence, $\alpha = 2\mu - 3.1308 - 5.28\delta^2$ with μ satisfying

$$\mu > \frac{1}{2}(3.1308 + 5.28\delta^2). \quad (77)$$

Then, if there is time-delay τ in TC and slave system S, then, by Theorem 3.3, for sufficiently large t, the synchronization error satisfies

$$\|e(t)\| \le \frac{\tau\|KH\|+1}{\frac{\alpha}{2}-\tau\|KH\|(\|A\|+L+\|KH\|)}\|\omega_\tau\|_\infty$$
$$= \frac{(\tau\mu+1)\|\omega_\tau\|_\infty}{-1.5654-2.64\delta^2+\mu[1-\tau(3.7322+2.64\delta^2+\mu)]}.$$

Moreover, by Corollary 3.1, the maximal time-delay satisfies

$$\tau^*_{\max} = \frac{(\sqrt{2}-1)^2}{\|A\|+L} = \frac{(\sqrt{2}-1)^2}{3.7322+2.64\delta^2}. \quad (78)$$

By [12], $\delta = 2.23$. Moreover, we calculate that $\mu^{**} = 40.6092, \tau^*_{\max} = 0.0102$. In simulations, without losing the generality, we set the initial conditions $x_i(t) = y_i(t)$, whenever $-\tau \le t < 0, i = 1,2,3$, and $[x_1(0) \quad x_2(0) \quad x_3(0) \quad y_1(0) \quad y_2(0) \quad y_3(0)] = [0.2\ 0.3\ 0.1\ 1.2\ 1.3\ 1.1]$, $\|\omega_\tau\|_\infty = 0.1$. Some noise is added. The noise is uniformly distributed on $[-\|\omega_\tau\|_\infty, \|\omega_\tau\|_\infty]$, generated by using $(\mathrm{rand}(1) - 0.5) \times 2 \times \|\omega_\tau\|_\infty$ in Matlab. First, let $\tau = 0.0102$, then, by Corollary 3.1, we get $\mu^* = ((\sqrt{2}-1))/(\tau) = 40.6092$, and for sufficiently large t, the synchronization error will satisfy $\|e(t)\| \le (\tau(\sqrt{2}+1))/((\sqrt{2}-1)-\tau(\|A\|+L)(\sqrt{2}+1))\|\omega_\tau\|_\infty = 0.5529\|\omega_\tau\|_\infty$. The attractors of the master and slave systems and the errors of synchronization are shown in Figs. 4–5, respectively, from which one can observe that all the synchronization errors are less than 0.02, consistent with the result obtained above: $\|e(t)\| \le 0.5529\|\omega_\tau\|_\infty = 0.05529$, based on Corollary 3.1. Then, letting $\tau = 0.0030$, then, by Corollary 3.1, we get $\mu^* = 138.0712$, and for sufficiently large t, the synchronization error will satisfy $\|e(t)\| \le 0.0891\|\omega_\tau\|_\infty$. The attractors of the master and slave systems and the errors of synchronization are shown in Figs. 6–7, respectively, from which one can observe that all the synchronization errors are less than 0.005, consistent with the result obtained above: $\|e(t)\| \le 0.0891\|\omega_\tau\|_\infty = 0.00891$, based on Corollary 3.1.

Remark 4.1: Examples 4.1 and 4.2 validate and demonstrate the theoretical results obtained in this paper. From Example 4.2, one can see that the estimation of synchronization errors and maximal time-delay indeed help the design of the linear control input to achieve uniform synchronization within a given error bound. Moreover, from Example 4.2, for the same TC noise $\omega(t)$, if we enlarge the control gain μ and reduce the time-delay τ, then, the smaller error $e(t)$ will be achieved.

Remark 4.2: By Remark 3.4, if the TC is noise-free, i.e., $\omega(t) = 0$, we can employ the Lyapunov–Krasovskii function in form of (62) to investigate the asymptotical synchronization. Since $\omega(t) = 0$, we set $r_4 = r_5 = 1$ in (64). From the expression of Φ in (64), it should be noticed that $\Phi < 0$ is not a LMI. In order to use LMI technique and make a comparison with the results in Example 4.2, we use the same controller designed in Example 4.2. Solving the LMI $\Phi < 0$ under these assumptions, we get

$$P = \begin{pmatrix} 2.6621 & 0.0135 & 0.0159 \\ 0.0135 & 2.6617 & 0.0108 \\ 0.0159 & 0.0108 & 2.6587 \end{pmatrix}$$

and the parameters $r_1 = 2.3577, r_2 = 2.5984, r_3 = 1.7468$, and the maximal time-delay $\tau^* = 0.0039$. Hence, by Remark 3.4, if the TC is noise-free, then, the asymptotical synchronization can be achieved within the maximal time-delay $\tau^* = 0.0039$. One can see from here that the maximal time-delay $\tau^*(= 0.0102)$ derived by using the Razumikhin technique in Example 4.2 is larger than that obtained by Lyapunov–Krasovskii function approach. Moreover, by using the Razumikhin technique, we can conclude that synchronization under noise-free is exponential and the synchronization error under the channel noise can also be easily estimated.

Remark 4.3: Here, we take Example 4.2 to make a comparison between Theorem 3.1 and Theorem 3.1*. In Example 4.2,

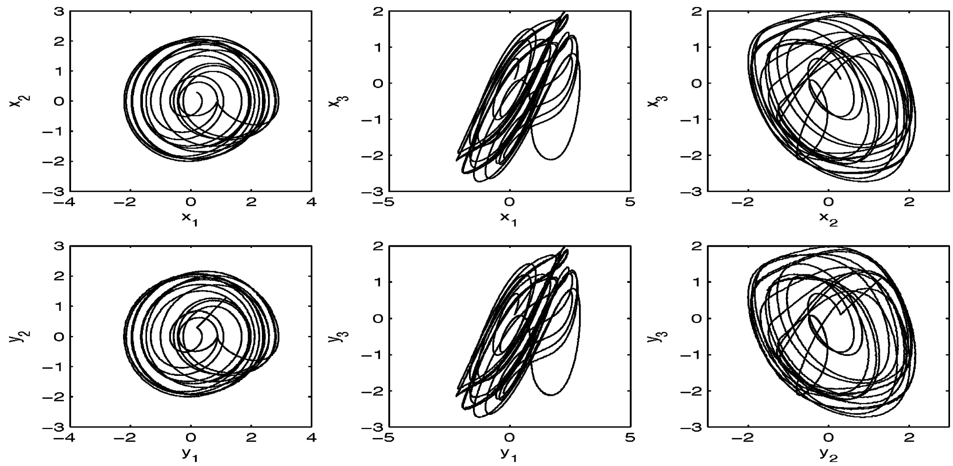

Fig. 4. Attractors of the master system and the slave system with $H = I, K = -40.6092I, \tau = 0.0102$, and $\|\omega_\tau\|_\infty = 0.1$.

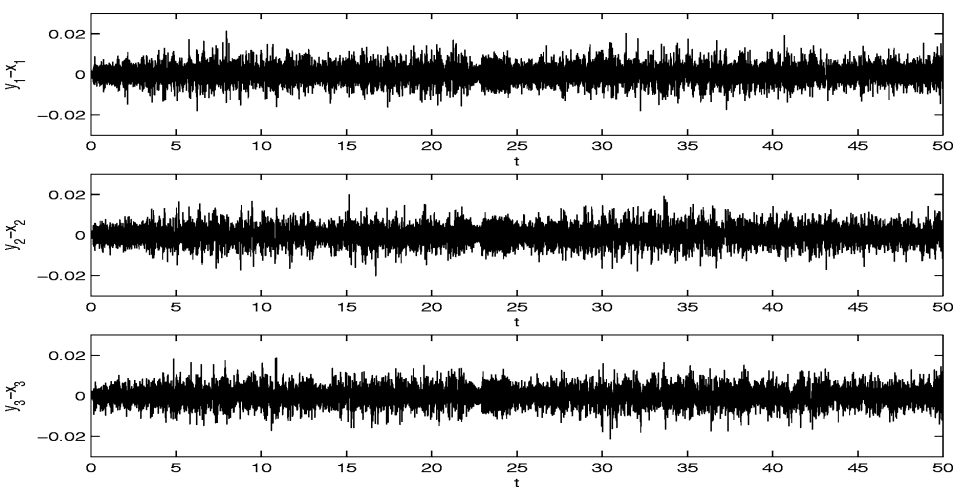

Fig. 5. Synchronization errors with $H = I, K = -40.6092I, \tau = 0.0102$, and $\|\omega_\tau\|_\infty = 0.1$.

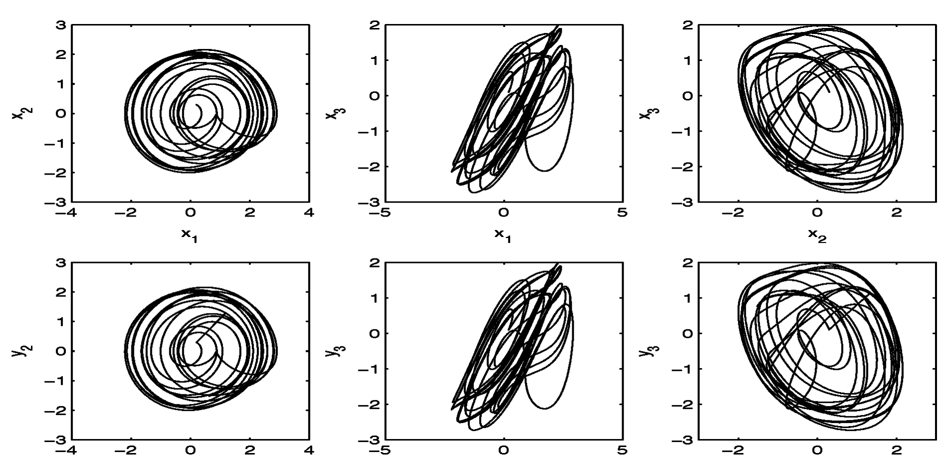

Fig. 6. Attractors of the master system and the slave system with $H = I, K = -138.0712I, \tau = 0.003$, and $\|\omega_\tau\|_\infty = 0.1$.

when the TC is noise-free, by (76)–(77), we get that the control gain coefficient μ must satisfy $\mu > 14.6939$ by Theorem 3.1, while $\mu > 10.7139$ by Theorem 3.1*. Moreover, for a given μ, we get the error estimation. For example, setting $\mu = 15$, then, for a sufficient large t, $\|e(t)\| \leq 3.2669\|\omega\|_\infty$ by Theorem 3.1, while $\|e(t)\| \leq 0.5018\|\omega\|_\infty$ by Theorem 3.1*. Hence, in Example 4.2, the results obtained by Theorem 3.1* is less conservative than that obtained by Theorem 3.1.

V. Conclusion

In this paper, global exponential synchronization and global uniform synchronization with an estimated error bound have

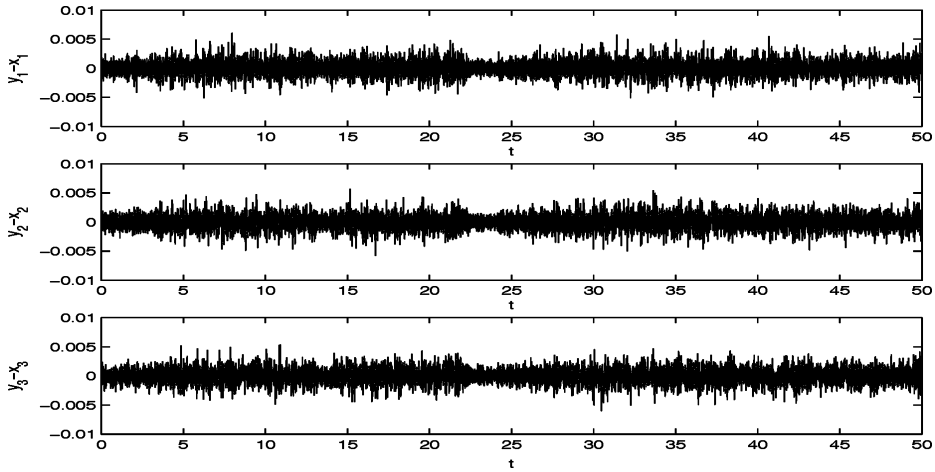

Fig. 7. Synchronization errors with $H = I, K = -138.0712I, \tau = 0.003$, and $\|\omega_\tau\|_\infty = 0.1$.

been investigated for the master–slave chaotic synchronization scheme via linear output control, possibly subject to unknown but bounded noise disturbances and time-delays in TC. By employing the methods of Lyapunov function, Razumikhin technique and ISS for nonlinear systems, estimation formulas for the synchronization error bounds have been derived. A maximal upper bound for time-delays was also estimated such that for any time-delay which is less than the maximal upper bound a linear output control is designed so that the synchronization error achieves the minimal value, even in the case with unknown but bounded noise disturbances and time-delays in TC. Meantime, a Razumikhin-type exponential stability theorem with Lyapunov exponent estimation for time-delay systems is also derived. The comparison between Razumikhin technique and Lyapunov–Krasovskii function method is also made. From theoretical analysis to numerical example, it shows that Razumikhin technique is efficient and less conservative. Two examples, namely, the chaotic Chua's circuit and Colpitts' oscillator, were presented, validating and also demonstrating the effectiveness of the theoretical results of the paper.

The methods and results obtained in this paper will be useful in the analysis and estimation of state errors for other kinds of systems such as network controlled systems and data-sampled systems, which is subject to disturbances by unknown but bounded channel noise and time-delays.

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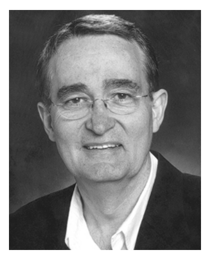

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